Proofs the properties of a homomorphism: These are in the book.

Chapter 10 #6. Let $P$ be the set of polynomials with real coefficients. $P$ is a group under addition. Define $\phi: P \to P$ by

$$\phi(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0) = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \ldots + \frac{a_1}{2} x + a_0 x + 0.$$  

Note that the constant term is zero because the problem requires the antiderivative to pass through $(0,0)$.

Let $f, g$ be two polynomials in $P$. Then it is easy to see that $\phi(f + g) = \phi(f) + \phi(g)$ hence that $\phi$ is a homomorphism.

$$Ker(\phi) = \{ f \in P | \phi(f) = 0 \} = 0 \text{ (Only the zero function maps to zero).}$$

If the mapping was defined so that the antiderivative passed through $(0,1)$ instead of $(0,0)$ then it would not be a homomorphism. Because then

$$\phi(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0) = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \ldots + \frac{a_1}{2} x + a_0 x + 1.$$  

For $x$ and $1$ in $P$ we have

$$\phi(x + 1) = \frac{x^2}{2} + x + 1 \text{ whereas } \phi(x) + \phi(1) = \frac{x^2}{2} + 1 + x + 1.$$  

Chapter 10 #8. For $sgn(\alpha \beta)$ there are four cases to consider:

case 1: $\alpha$ even $\beta$ even.
case 2: $\alpha$ even $\beta$ odd
case 3: $\alpha$ odd $\beta$ even
case 4: $\alpha$ odd $\beta$ odd

Check that in either case the operation is preserved.

$$Ker(sgn) = \{ \sigma \in S_\alpha | sgn(\sigma) = 1 \} = A_\alpha.$$  

Chapter 10 #17. Suppose there is a homomorphism $\phi$ from $\mathbb{Z}_{16} \otimes \mathbb{Z}_2$ onto $\mathbb{Z}_4 \otimes \mathbb{Z}_4$. Then by the first isomorphism theorem

$$\mathbb{Z}_{16} \otimes \mathbb{Z}_2 / Ker\phi \cong \mathbb{Z}_4 \otimes \mathbb{Z}_4.$$  

Comparing the number of elements we get that $|Ker\phi| = 2$. Since $\mathbb{Z}_{16} \otimes \mathbb{Z}_2$ is Abelian, all subgroups are normal. So possibilities for $Ker\phi$ are subgroups of order 2 in $\mathbb{Z}_{16} \otimes \mathbb{Z}_2$. There are three such subgroups:
\( H_1 = \{(0,0),(8,1)\}, \ H_2 = \{(0,0),(0,1)\}, \) and \( H_3 = \{(0,0),(8,0)\} \).

If \( \text{Ker}\phi = H_1 \) then \((1,0) + \text{Ker}\phi \) has order 16 but \( \mathbb{Z}_4 \otimes \mathbb{Z}_4 \) has no element of order 16.
If \( \text{Ker}\phi = H_2 \) then \((1,0) + \text{Ker}\phi \) has order 16 but \( \mathbb{Z}_4 \otimes \mathbb{Z}_4 \) has no element of order 16.
If \( \text{Ker}\phi = H_3 \) then \((1,0) + \text{Ker}\phi \) has order 8 but \( \mathbb{Z}_4 \otimes \mathbb{Z}_4 \) has no element of order 8.
Therefore there are no homomorphisms from \( \mathbb{Z}_{16} \otimes \mathbb{Z}_2 \) onto \( \mathbb{Z}_4 \otimes \mathbb{Z}_4 \).

Chapter 10 #20. There are no homomorphisms from \( \mathbb{Z}_{20} \) onto \( \mathbb{Z}_8 \) because if there were such a homomorphism, say \( \phi \), then by the first isomorphism theorem \( \mathbb{Z}_{20}/\text{Ker}\phi \cong \mathbb{Z}_8 \). This would imply \( 20 = |\text{Ker}\phi| \times 8 \) (why?) which is not possible.

There are homomorphisms from \( \mathbb{Z}_{20} \) to \( \mathbb{Z}_8 \) that are not onto.
Let \( \phi \) be one such map. Then \( \mathbb{Z}_{20}/\text{Ker}\phi \cong \text{Im}\phi \). Since \( \mathbb{Z}_{20} \) is finite, we have \( |\text{Ker}\phi||\text{Im}\phi| = 20 \).

\( \text{Ker}\phi \) is a subgroup of \( \mathbb{Z}_{20} \) and \( \text{Im}\phi \) is a subgroup of \( \mathbb{Z}_8 \) so possibilities are
1. \( |\text{Ker}\phi| = 5 \) and \( |\text{Im}\phi| = 4 \)
2. \( |\text{Ker}\phi| = 10 \) and \( |\text{Im}\phi| = 2 \)

Recall that \( \mathbb{Z}_{20} \) has only one subgroup of order 5, and only one subgroup of order 10.
Because \( \mathbb{Z}_{20} \) is cyclic \( \phi \) is determined by its action on its generator 1.

For case 1, i.e. when \( |\text{Ker}\phi| = 5 \), we have \( \text{Ker}\phi = \{0,4,8,12,16\} \).
Let \( \phi(1) = 2 \) (it should map to something of order 4 in \( \mathbb{Z}_8 \), why?).

All the elements in a coset should map to the same element (why?), so elements in \( 0 + \text{Ker}\phi = \{0,4,8,12,16\} \) map to 0.
elements in \( 1 + \text{Ker}\phi = \{1,5,9,13,17\} \) map to 2.
elements in \( 2 + \text{Ker}\phi = \{2,6,10,14,18\} \) map to 4. (why?)
elements in \( 3 + \text{Ker}\phi = \{3,7,11,15,19\} \) map to 6. (why?)

Carry out a similar argument for the second case, that is when \( |\text{Ker}\phi| = 10 \).

Chapter 10 #33. \( \phi : U_{40} \to U_{40} \) is a homomorphism with kernel \( \{1,9,17,33\} \). Suppose \( \phi(11) = 11 \)
Note that for a homomorphism \( \phi \), \( \phi(a) = \phi(b) \) iff \( \phi(ab^{-1}) = 1 \) iff \( ab^{-1} \in \text{Ker}\phi \), that is \( a \in b\text{Ker}\phi \). Therefore if \( \phi(11) = 11 \) then the elements that map to 11 are precisely the elements in \( 11\text{Ker}\phi = \{11,19,27,3\} \).

Chapter 10 #34. Find a homomorphism as in problem 33.
We need to determine where each of the cosets of \( \text{Ker}\phi \) map. We already have elements in \( 11\text{Ker}\phi = \{1,9,17,33\} \) map to 1.
elements in \( 11\text{Ker}\phi = \{11,19,27,3\} \) map to 11.
If we map the other two cosets as follows we get a homomorphism as desired:
Elements in $7\ker \phi = \{7, 23, 39, 31\}$ map to 7.
Elements in $13\ker \phi = \{13, 37, 21, 29\}$ map to 13.

Think: Can you find others?

Chapter 10 #36. Let $\phi : \mathbb{Z} \otimes \mathbb{Z} \to G$ be such that $\phi((3, 2)) = a$ and $\phi((2, 1)) = b$.
Then $\phi((3, 2) - (2, 1)) = \phi((3, 2)) - \phi((2, 1)) = a - b$ and $\phi((4, 4)) = \phi(4(1, 1)) = 4\phi((1, 1)) = 4(a - b)$ using the fact that $\phi$ is a homomorphism.

Chapter 10 #39. Second Isomorphism Theorem: If $K$ is a subgroup of $G$ and $N$ is a normal subgroup of $G$, then $K/(K \cap N) \cong KN/N$.

1. Prove that $K \cap N$ is normal in $K$.
2. Prove that $N$ is normal in $KN$.
3. Prove that the map $\phi : K \to KN/N$ given by $\phi(k) = kN$ is a homomorphism and is onto.
4. Apply the first isomorphism theorem.

Proofs:
1. Let $a \in K \cap N$ and $b \in K$. We need to prove that $x = bab^{-1} \in K \cap N$. Since $N$ is normal in $G$ and $b \in G$ and $a \in N$ we have $x \in N$.
   $x$ is also in $K$ because $a$ and $b$ are both in $K$ and $K$ is a subgroup.

2. Let $a \in N$ and $b \in KN$. We need to prove that $x = bab^{-1} \in N$. Since $N$ is normal in $G$ and $b \in KN \subset G$ and $a \in N$ we have $x \in N$.

3. Let $k_1, k_2 \in K$. Then $\phi(k_1k_2) = (k_1k_2)N = k_1Nk_2N = \phi(k_1)\phi(k_2)$. The second equality holds by definition of product of cosets. Therefore $\phi$ is a homomorphism.

   $\phi$ is onto because a coset in $KN/N$ is of the form $knN$ for some $k \in K$ and $n \in N$. Note that $nN = N$ because $n \in N$. Then $\phi(K) = kN$ so any coset coset in $KN/N$ is hit by $\phi$.

4. $\ker \phi = \{k \in K | \phi(k) = N\}$ so $\ker \phi = \{k \in K | kN = N\} = \{k \in K | k \in N\} = K \cap N$.
Hence by the first isomorphism theorem $K/(K \cap N) \cong KN/N$

Chapter 10 #50. Let $G = \langle a \rangle$. Any element in $G$ is of the form $a^k$ for some $k \in \mathbb{Z}$ and for a homomorphism $\phi$, $\phi(a^k) = (\phi(a))^k$. So if $\phi(a)$ is given, the images of the rest of the elements will be determined by it.