Math 229 – Fall 2020 Distributed Tuesday, August 25

Problem Set 1

1. Say we have 12 coins, one of which is counterfeit and either heavier or lighter than the rest **(we don't know whether it's heavier or lighter)**. We have a balance scale. Describe how in three weighings, one can determine which coin is the counterfeit and whether it is heavier or lighter than the rest. (This problem isn't really a contest problem, but it is a classic.)

2. (VTRMC) The number $2^{48} - 1$ is exactly divisible by what two numbers between 60 and 70? (Remember, calculators are not allowed.)

3. For each positive integer n, let S_n denote the total number of squares in an $n \times n$ grid. Thus $S_1 = 1$ and $S_2 = 5$, because a 2×2 square grid has four 1×1 squares and one 2×2 square. Find a recurrence relation for S_n, and use it to calculate the total number of squares on a chessboard (i.e. determine S_8).

4. (IMO 1972) Prove that from ten distinct two-digit numbers, one can always choose two disjoint, nonempty subsets, so that their elements have the same sum.

5. Find, and give a proof of your answer, all positive integers n such that neither n nor n^2 contain a 1 when written in base 3.

6. (Another classic) A domino will exactly cover two adjacent squares on a chessboard. Hence, a chessboard may be covered using 32 dominoes. If squares at two opposite corners of a chessboard are removed, can the remaining squares be covered by 31 dominoes?

7. Describe all functions $f : \mathbf{N} \to \mathbf{N}$ with the property that

$$f(mn) = f(m)f(n)$$

for all $m, n \in \mathbf{N}$.