

GRE Prep - Linear and Abstract Algebra

1. Two distinct solutions \mathbf{x}_1 and \mathbf{x}_2 can be found to the linear system $A\mathbf{x} = \mathbf{b}$. Which of the following is necessarily true?

- (A) $\mathbf{b} = 0$.
- (B) A is invertible.
- (C) A has more columns than rows.
- (D) $\mathbf{x}_1 = -\mathbf{x}_2$.
- (E) There exists a solution \mathbf{x} such that $\mathbf{x} \neq \mathbf{x}_1$, and $\mathbf{x} \neq \mathbf{x}_2$.

2. Let A , B , and C be real 2×2 matrices, and let 0 denote the 2×2 zero matrix. Which of the following statements is/are true?

- I. $A^2 = 0 \Rightarrow A = 0$
- II. $AB = AC \Rightarrow B = C$
- III. A is invertible and $A = A^{-1} \Rightarrow A = I$ or $A = -I$

- (A) I only
- (B) I and III only
- (C) II and III only
- (D) III only
- (E) None of the above

3. If the matrices

$$\begin{pmatrix} 3 & -2 & -2 \\ -1 & 1 & 1 \\ 3 & -1 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & a & 0 \\ -1 & b & 1 \\ 2 & c & -1 \end{pmatrix}$$

are inverses of each other, what is the value of c ?

- (A) -3
- (B) -2
- (C) 0
- (D) 2
- (E) 3

4. The linear transformation $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ that maps $(1, 2)$ to $(-1, 1)$ and $(0, -1)$ to $(2, -1)$ will map $(1, 1)$ to

- (A) $(1, 2)$
- (B) $(1, 0)$
- (C) $(2, -1)$
- (D) $(2, 1)$
- (E) $(1, 1)$

5. Define linear operators S and T on the xy -plane (\mathbf{R}^2) as follows: S rotates each vector 90° counterclockwise, and T reflects each vector through the y -axis. If ST and TS denote the compositions $S \circ T$ and $T \circ S$ respectively, and I is the identity map, which of the following is true?

- (A) $ST = I$
- (B) $ST = -I$
- (C) $TS = I$
- (D) $ST = TS$
- (E) $ST = -TS$

6. Which one of the following groups is cyclic?
- (A) $\mathbf{Z}_2 \times \mathbf{Z}_4$
 - (B) $\mathbf{Z}_2 \times \mathbf{Z}_6$
 - (C) $\mathbf{Z}_3 \times \mathbf{Z}_4$
 - (D) $\mathbf{Z}_3 \times \mathbf{Z}_6$
 - (E) $\mathbf{Z}_4 \times \mathbf{Z}_6$
7. If G is a group of order 12, then G must have a subgroup of all the following orders EXCEPT
- (A) 2
 - (B) 3
 - (C) 4
 - (D) 6
 - (E) 12
8. Which of the following are subgroups of $GL(2, \mathbf{R})$, the group of invertible 2 by 2 matrices (with real entries) under matrix multiplication?
- I. $T = \{A \in GL(2, \mathbf{R}) : \det(A) = 2\}$
 - II. $T = \{A \in GL(2, \mathbf{R}) : A \text{ is upper triangular}\}$
 - III. $T = \{A \in GL(2, \mathbf{R}) : \text{tr}(A) = 0\}$
- (A) I and II only
 - (B) II only
 - (C) II and III only
 - (D) III only
 - (E) I and III only
9. Let H be the set of all group homomorphisms $\phi : \mathbf{Z}_3 \rightarrow \mathbf{Z}_6$. How many functions does H contain?
- (A) 1
 - (B) 2
 - (C) 3
 - (D) 4
 - (E) 6
10. How many generators does the group $(\mathbf{Z}_{24}, +)$ have?
- (A) 2
 - (B) 6
 - (C) 8
 - (D) 10
 - (E) 12