MATH 150 HW 7 SOLUTIONS

Section 8.7 #18. $\lim_{x\to 0^+} \frac{e^x-(1+x)}{x^n} \to \frac{0}{0} \Rightarrow$ Apply L'Hospitals

$= \lim_{x\to 0^+} \frac{e^x-1}{nx^{n-1}} \to$ if $n=1$ then we have $\lim_{x\to 0^+} \frac{e^x-1}{1} = 0$.

if $n>1$ then we have $\lim_{x\to 0^+} \frac{e^x-1}{nx^{n-1}} \to \frac{0}{0}$

So apply L'Hospital again, $\lim_{x\to 0^+} \frac{e^x}{n(n-1)x^{n-2}}$ → if $n=2$ then limit is $\frac{1}{2}$

if $n>2$ then $\frac{1}{0} = \infty$.

#28. $\lim_{x\to\infty} \frac{x^2}{e^x} \to \frac{\infty}{\infty} \xRightarrow{\text{L'Hospital}}$ the limit is $= \lim_{x\to\infty} \frac{2x}{e^x} \to \frac{\infty}{\infty}$

L'Hospital again $\Rightarrow$ limit is $= \lim_{x\to\infty} \frac{2}{e^x} \to \frac{2}{\infty} = \boxed{0}$.

#46. $\lim_{x\to\infty} (1+x)^{1/x} \to \infty^0$ indeterminate

let $y = \lim_{x\to\infty} (1+x)^{1/x}$

Take ln of both sides $\Rightarrow$ $\ln y = \lim_{x\to\infty} \frac{1}{x}\cdot \ln(1+x)$

$= \lim_{x\to\infty} \frac{\ln(1+x)}{x} \to \frac{\infty}{\infty}$

L'Hospital $\Rightarrow$ $= \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} = 0$.

So, $\ln y = 0$. Then $\boxed{y = e^0 = 1}$

Section 8.8 #20 $\int_0^\infty xe^{-x/2}\,dx = \lim_{a\to\infty} \int_0^a xe^{-x/2}\,dx$

integration by parts

$u = x \qquad dv = e^{-x/2}dx$

$du = dx \qquad v = -2e^{-x/2}$

$\int_0^a xe^{-x^2}dx = x\cdot(-2e^{-x/2}) - \int -2e^{-x/2}dx$

$= -2xe^{-x/2} + 2e^{-x/2}\cdot(-2)$

$= -2xe^{-x/2} - 4e^{-x/2}\Big|_0^a = -2ae^{-a/2} - 4e^{-a/2} - (0 - 4e^{-0/2})$

$= -e^{-a/2}(2a+4) + 4$

$\lim_{a\to\infty} - \frac{2a+4}{e^{a/2}} + 4 = 4$

Section 8-8 #36. $\displaystyle\int_0^6 \frac{4}{\sqrt{6-x}}\,dx = \lim_{a\to 6^-} \int_0^a \frac{4}{\sqrt{6-x}}\,dx$

$$= \lim_{a\to 6^-} 4\cdot\int_0^a (6-x)^{-1/2}\,dx = \lim_{a\to 6^-} 4\cdot \frac{(6-x)^{1/2}}{1/2}\cdot\frac{1}{(-1)}\Bigg|_0^a = \lim_{a\to 6^-} -8(6-x)^{1/2}\Big]_0^a$$

$$= \lim_{a\to 6^-}\left(-8\,\cancelto{0}{\sqrt{6-a}} - \left(-8\sqrt{6-0}\right)\right) = 8\sqrt{6}$$

#70. Witch of Agnesi $\quad y = \dfrac{8}{x^2+4}$

Because the graph is symmetric the area is $2\displaystyle\int_0^\infty \frac{8}{x^2+4}\,dx$.

Then

$$\text{Area} = 16\int_0^\infty \frac{1}{x^2+4}\,dx = 16\cdot\lim_{a\to\infty}\int_0^a \frac{1}{x^2+4}\,dx$$

$$16\cdot\lim_{a\to\infty} \frac{1}{2}\arctan\frac{x}{2}\Bigg|_0^a = 16\lim_{a\to\infty}\left(\cancelto{\pi/2}{\frac{1}{2}\arctan\frac{a}{2}} - \cancelto{0}{\frac{1}{2}\arctan\frac{0}{2}}\right)$$

$$= 16\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 4\pi.$$

Section 9-1 #8. $a_n = (-1)^{n+1}\left(\frac{2}{n}\right)$

$a_1 = 2,\ a_2 = -1,\ a_3 = \frac{2}{3},\ a_4 = -\frac{1}{2},\ a_5 = \frac{2}{5}$

#28. $\underset{a_1}{1},\ \underset{a_2}{-\frac{1}{2}},\ \underset{a_3}{\frac{1}{4}},\ \underset{a_4}{-\frac{1}{8}}\ \dots \qquad a_n = (-1)^{n+1}\dfrac{1}{2^{(n-1)}}$

#40. $\displaystyle\lim_{n\to\infty}\frac{5n}{\sqrt{n^2+4}}$. Consider $\displaystyle\lim_{x\to\infty}\frac{5x}{\sqrt{x^2+4}} = \lim_{x\to\infty}\frac{5x}{\sqrt{x^2}} = 5$

dominant terms in the numerator & denominator

So, $\displaystyle\lim_{n\to\infty}\frac{5n}{\sqrt{n^2+4}} = 5$.

#42. $\displaystyle\lim_{n\to\infty}\cos\frac{2}{n}$. Consider $\displaystyle\lim_{x\to\infty}\cos\frac{2}{x} \to \cos\frac{2}{\infty} \to \cos 0 \to 1$

So, $\displaystyle\lim_{n\to\infty}\cos\frac{2}{n} = \underline{1}$.

#48. $a_n = 1+(-1)^n \Rightarrow 0, 1, 0, 1, 0, 1, 0, 1 \cdot$ limit does not exist.

#52. $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} = \frac{1 \cdot 3 \cdot 5 \cdots 2n-1}{1 \cdot 2 \cdot 3 \cdots n}$

$= 1 \cdot \frac{3}{2} \cdot \frac{5}{3} \cdot \frac{7}{4} \cdots \frac{2n-1}{n} \geqslant 1 \cdot \underbrace{\frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdots \frac{3}{2}}_{n-1 \text{ times}}$

$\Rightarrow a_n \geqslant \left(\frac{3}{2}\right)^{n-1}$.

Since $\lim_{n\to\infty} \left(\frac{3}{2}\right)^{n-1}$ is ∞ and $\lim_{n\to\infty} a_n$ should be greater than or equal to that, so $\boxed{a_n \text{ is divergent}}$

#62. $\lim_{n\to\infty} \left(\frac{n^2}{2n+1} - \frac{n^2}{2n-1}\right) = \lim_{n\to\infty} \frac{n^2(2n-1) - n^2(2n+1)}{(2n+1)(2n-1)}$ (get a common denominator)

$= \lim_{n\to\infty} \frac{-2n^2}{4n^2-1} = \frac{-2}{4} = \boxed{\frac{-1}{2}}$