Section 9.2 #14. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$ is divergent because if you look at the n-th term, $\frac{n}{\sqrt{n^2+1}}$, it goes to 1 as n goes to ∞.

#24. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Partial fractions. $\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$

$\frac{1}{n(n+2)} = \frac{A(n+2)+Bn}{n(n+2)} \Rightarrow 1 = 2A + (A+B)n$

$\Downarrow$

$A = \frac{1}{2}$, $A+B=0$ $\Rightarrow B = \frac{-1}{2}$

So, $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2n} - \frac{1}{2(n+2)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$

$= \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6} + \cdots\right)$

Look at kth sum: $\frac{1}{2}\sum_{n=1}^{k} \frac{1}{n} - \frac{1}{n+2} = \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \frac{1}{6} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} + \cdots + \cancel{\frac{1}{k-1}} - \frac{1}{k+1} + \cancel{\frac{1}{k}} - \frac{1}{k+2}\right)$

$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}\right)$

$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n(n+2)} = \lim_{k\to\infty} \frac{1}{2}\left(1 + \frac{1}{2} - \cancelto{0}{\frac{1}{k+1}} - \cancelto{0}{\frac{1}{k+2}}\right) = \frac{1}{2}\cdot\frac{3}{2} = \frac{3}{4}$

#26. $\sum_{n=1}^{\infty} 2\left(-\frac{1}{2}\right)^n \Rightarrow$ geometric series with common ratio $r = -\frac{1}{2}$

Since $-1 < -\frac{1}{2} < \frac{1}{2}$, the series is convergent

#38. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$ (hint. use partial fractions) and proceed as in #24 above.

$\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2n+1} - \frac{1}{2n+3} = \lim_{k\to\infty} \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2k+3}\right) = \frac{1}{6}$

#40. $\sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n$

$\Downarrow$ geometric series with $r = \frac{4}{5}$ $= 6\left(1 + r + r^2 + r^3 + \cdots\right) = \frac{6}{1-r}$

So the sum is $\frac{6}{1-\frac{4}{5}} = 30$

#42. $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n = 2\left(1 + r + r^2 + r^3 + \cdots\right) = \frac{2}{1-r}$

$\hookrightarrow$ geometric series with $r = -\frac{2}{3}$

So the sum is $\frac{2}{1-\left(-\frac{2}{3}\right)} = \frac{6}{5}$

Section 9.2 # 62. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ divergent because $\lim_{n\to\infty} \frac{3^n}{n^3} = \infty$ (why?)

(nth term test says if $\lim_{n\to} a_n \neq 0$ then $\sum a_n$ is divergent)

#68. $\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$ divergent because $\lim_{n\to\infty} \ln\left(\frac{1}{n}\right) \to \ln\left(\frac{1}{\infty}\right) \to \ln(0) \to -\infty \to$ not zero! $\to$ series is divergent

#72. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \ln(n+1) - \ln(n) = \ln 2 - \ln 1 + \ln 3 - \ln 2 + \cdots$

(what does the nth term test say?)

$$= \lim_{k\to\infty} \sum_{n=1}^{k} (\ln(n+1) - \ln n)$$

$$= \lim_{k\to\infty} \cancel{\ln 2} - \ln 1 + \cancel{\ln 3} - \cancel{\ln 2} + \ln 4 - \cancel{\ln 3} + \cdots + \ln(k+1) - \cancel{\ln k}$$

$$= \lim_{k\to\infty} \left(-\ln 1 + \ln(k+1)\right) = \infty .$$

The sum is infinite so the series is divergent.

#77. Let $a_n = \frac{n+1}{n}$. Discuss the convergence of $\{a_n\}$ $\sum_{n=1}^{\infty} a_n$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

Since a_n does not go to zero, $\sum_{n=1}^{\infty} a_n$ is divergent