Section 9.6 #28. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{[(n+1)!]^2}{[3(n+1)]!}}{\frac{(n!)^2}{(3n)!}}\right| = \frac{(n+1)\cdot n!\ (n+1)\cdot \cancel{n!}}{(3n+3)!} \cdot \frac{(3n)!}{n!\cdot \cancel{n!}} = \frac{(n+1)^2 \cancel{(3n)!}}{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} \Rightarrow \frac{n^2}{27n^3} \to 0 \text{ as } n \text{ goes to } \infty.$$

By ratio test, since $0<1$, the series is convergent.

#48. $\sum_{n=1}^{\infty} \left(\frac{\ln}{n}\right)^n$ Root Test: $\lim_{n\to\infty} \sqrt[n]{\left(\frac{\ln(n)}{n}\right)^n} = \lim_{n\to\infty} \frac{\ln n}{n} \to \frac{\infty}{\infty}$

L'Hospital on $\lim_{x\to\infty} \frac{\ln x}{x} \to \frac{\frac{1}{x}}{1} = 0$. Hence $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

Since $0<1$, by root test the series is convergent.

Section 9.8 #14. $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+2}\ (n+2)\ x^{n+1}}{(-1)^{n+1}\ (n+1)\ x^n}\right| = \left|\frac{(n+2)}{(n+1)} x\right|$

$\lim_{n\to\infty} \left|\frac{(n+2)}{(n+1)} x\right| = |x|$

goes to 1

If $|x|<1$, $\sum(-1)^{n+1}(n+1)x^n$ is convergent

$-1<x<1 \Rightarrow$ $\boxed{\text{radius of convergence is } 1}$

Check the end points for interval of convergence:

plug in $x=-1$ in the series $\Rightarrow \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)(-1)^n = \sum_{n=0}^{\infty}(-1)^{2n+1}(n+1) = \sum_{n=0}^{\infty} -(n+1)$

odd number

always negative

$\sum -(n+1)$ is divergent because $\lim_{n\to\infty} -(n+1) = -\infty \neq 0$.

plug in $x=1$ in the series $\Rightarrow \sum(-1)^{n+1}(n+1)(1)^n = \sum(-1)^{n+1}(n+1)$

$\sum(-1)^{n+1}(n+1)$ is divergent because $\lim_{n\to\infty} (-1)^{n+1}(n+1) \neq 0$

So, interval of convergence is $\boxed{(-1,1)}$

Section 9.8 #20. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, n!\, (x-4)^n}{3^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}\,(n+1)!\,(x-4)^{n+1}}{3^{n+1}}}{\frac{(-1)^n\, n!\, (x-4)^n}{3^n}}\right| = \left|\frac{n+1}{3}(x-4)\right|$$

$\displaystyle\lim_{n\to\infty}\left|\frac{n+1}{3}(x-4)\right| \to \infty$ unless $x-4=0$, and when $x=4$ the series is $\sum 0 = 0$

So the series converges at a single point, namely at $x=4$.

#28. $\displaystyle\sum \frac{(-1)^n x^{2n}}{n!}$ $\quad \displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1} x^{2(n+1)}}{(n+1)!}}{\frac{(-1)^n x^{2n}}{n!}}\right| = \left|\frac{x^{2n+2}}{(n+1)!}\cdot\frac{n!}{x^{2n}}\right| = \left|\frac{x^2}{n+1}\right|$

$\displaystyle\lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0$ for any x.

So by ratio test, the series converges for all real numbers x.

#48. a) $\displaystyle f(x) = \sum (-1)^{n+1}\frac{(x-2)^n}{n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+2}(x-2)^{n+1}}{n+1}\cdot\frac{n}{(-1)^{n+1}\cdot(x-2)}\right| = \left|\frac{n}{n+1}(x-2)\right|$$

$\displaystyle\lim_{n\to\infty}\left|\frac{n}{n+1}(x-2)\right| = |x-2|$, $\quad |x-2|<1 \Rightarrow$ convergent $\Rightarrow$
$-1 < x-2 < 1 \Rightarrow 1 < x < 3$.

Check the end points: plug in $x=1 \Rightarrow \displaystyle\sum \frac{(-1)^{n+1}(-1)^n}{n} = \sum -\frac{1}{n} = -\underbrace{\sum \frac{1}{n}}_{\text{divergent (harmonic series)}}$

plug in $x=3 \Rightarrow \displaystyle\sum \frac{(-1)^{n+1} 1^n}{n} = \underbrace{\sum \frac{(-1)^{n+1}}{n}}_{\text{convergent, alternating harmonic series}}$

So interval of convergence is
$1 < x \le 3$.

b)
$$f'(x) = \sum \frac{(-1)^{n+1}\, \cancel{n}\, (x-2)^{n-1}}{\cancel{n}} = \sum (-1)^{n+1}(x-2)^{n-1}$$

For interval of convergence just check the end points $x=1$ & $x=3$
you'll see that interval of convergence is $1 < x < 3$

section 9.8 #48 continued

c) $f''(x) = \sum (-1)^{n+1} \dfrac{n \cdot (n-1)(x-2)^{n-2}}{n} = \sum (-1)^{n+1} (n-1)(x-2)^{n-2}$

Again not convergent at $x=1$ or $x=3$ so interval of convergence is $1 < x < 3$.

d) $\int f(x)\,dx = \sum \dfrac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)}$

Check end points $x=1$ and $x=3$ for interval of convergence. You'll see that in both cases the series is convergent so interval of convergence is $1 \le x \le 3$.

#73. $f(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$

Approximate $f(x)$ using the first 10 terms.

$$\sum_{n=0}^{9} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \quad - \frac{x^{18}}{18!}$$

(see the DERIVE file)

#74 $f(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$

$$\sum_{n=0}^{9} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots - \frac{x^{19}}{19!}$$

(see the DERIVE file)