Math 360 ALGEBRA HOMEWORK 8 (EXTRA) SOLUTIONS

Chapter 9 #4. Is $H = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ a normal subgroup of $GL(2, \mathbb{R})$?

No. For example, let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in H$.

Then $ABA^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ which is not in H, so H is not normal in G.

Chapter 9 #10. Prove that a factor group of a cyclic group is cyclic.

Let $G = \langle a \rangle$.

Claim: G/H is generated by aH.
Let $bH \in G/H$. (We need to show that $bH = (aH)^t$ for some $t \in \mathbb{Z}$.)
Since $b \in G$, $b = a^r$ for some $r \in \mathbb{Z}$. Then $(aH)^r = a^r H = bH$ and we are done.

Chapter 9 #14. What is the order of $14 + \langle 8 \rangle$ in $\mathbb{Z}_{24}/\langle 8 \rangle$?

Note that $\langle 8 \rangle = \{0, 8, 16\}$. We have

$2\,(14 + \langle 8 \rangle) = (14 + \langle 8 \rangle) + (14 + \langle 8 \rangle) = 4 + \langle 8 \rangle$ because $14 + 14 \equiv 4 \mod 24$,
$3\,(14 + \langle 8 \rangle) = (4 + \langle 8 \rangle) + (14 + \langle 8 \rangle) = 18 + \langle 8 \rangle$
$4\,(14 + \langle 8 \rangle) = (18 + \langle 8 \rangle) + (14 + \langle 8 \rangle) = 8 + \langle 8 \rangle$ because $18 + 14 \equiv 8 \mod 24$.
Since $8 + \langle 8 \rangle = \langle 8 \rangle =$ identity in $\mathbb{Z}_{24}/\langle 8 \rangle$, order of $14 + \langle 8 \rangle$ is 4.

Chapter 9 #38. Let H be a normal subgroup of G and let $a \in G$. If the element aH has order 3 in the group G/H and $|H| = 10$, what are the possibilities for the order of a?

$|aH| = 3$ implies that $a^3 \in H$. Then the order of a^3 divides 10 by Lagrange's Theorem. So possibilities for order of a^3 are 1,2,5,or 10.

That is, $a^3 = e$ or $a^6 = e$ or $a^1 5 = e$ or $a^3 0 = e$. So possible orders of a are 3,6,15, or 30. (Think about why it can't be 1,2,or 5.)

Chapter 9 #44. If $|G| = pq$, where p and q are primes that are not necessarily distinct, prove that $|Z(G)| = 1$ or pq.

Suppose $Z(G) \neq 1$. Then $Z(G)$ contains elements besides identity.

We will prove that $|Z(G)| = pq$.

For sake of contradiction, suppose $|Z(G)| \neq pq$. Then $Z(G) \neq G$, that is, there exists an $a \in G, a \notin Z(G)$.

Recall that centralizer of a in G, $C(a)$ is a subgroup of G and it contains $Z(G)$. It also contains $< a >$.

Since $a \notin Z(G)$, we have $|Z(G)| < |C(a)| \leq pq$. Since $Z(G) = p$ or q by lagrange's Theorem and our assumptions above, and since again by Lagrange's Theorem $|C(G)$ divides $|G|$, we get $C(G) = G$. That is a commutes with every element in G which means $a \in Z(G)$. But this contradicts with $a \notin Z(G)$. Therefore $|Z(G)| = pq$.

Chapter 9 #48. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k, jk = -kj = i, ki = -ki = j$, $-i = (-1)i, -j = (-1)j, -k = (-1)k$.

a) Cayley table for G.

	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

b) $H = \{1, -1\}$ normal in G.

Note that $a1a^{-1} = 1$ for any $a \in G$ and $a(-1)a^{-1} = (-1)$ for any $a \in G$. So H is normal in G.

c) Cayley table for G/H.

Note that G/H has 8/2=4 elements.

$H = \{1, -1\}$
$iH = \{i, -i\}$
$jH = \{j, -j\}$
$kH = \{k, -k\}$

Call these E, I, J,and K respectively.

	E	I	J	K
E	E	I	J	K
I	I	E	K	J
J	J	K	E	I
K	K	J	I	E