Math 360 ALGEBRA HOMEWORK 9 SOLUTIONS

Proofs the properties of a homomorphism: These are in the book.

Chapter 10 #6. Let P be the set of polynomials with real coefficients. P is a group under addition. Define $\phi : P \to P$ by

$$\phi(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0) = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \ldots + \frac{a_1}{2}x + a_0 x + 0.$$

Note that the constant term is zero because the problem requires the antiderivative to pass through (0,0).

Let f, g be two polynomials in P. Then it is easy to see that $\phi(f+g) = \phi(f) + \phi(g)$ hence that ϕ is a homomorphism.

$Ker(\phi) = \{f \in P | \phi(f) = 0\} = 0$ (Only the zero function maps to zero).

If the mapping was defined so that the antiderivative passed through (0,1) instead of (0,0) then it would not be a homomorphism. Because then

$$\phi(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0) = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \ldots + \frac{a_1}{2}x + a_0 x + 1.$$

For x and 1 in P we have

$\phi(x+1) = \frac{x^2}{2} + x + 1$ whereas $\phi(x) + \phi(1) = \frac{x^2}{2} + 1 + x + 1$.

Chapter 10 #8. For $sgn(\alpha\beta)$ there are four cases to consider:
case 1: α even β even.
case 2: α even β odd
case 3: α odd β even
case 4: α odd β odd

Check that in either case the operation is preserved.

$Ker(sgn) = \{\sigma \in S_n | sgn(\sigma) = 1\} = A_n$.

Chapter 10 #17. Suppose there is a homomorphism ϕ from $\mathbb{Z}_{16} \otimes \mathbb{Z}_2$ onto $\mathbb{Z}_4 \otimes \mathbb{Z}_4$. Then by the first isomorphism theorem

$$\mathbb{Z}_{16} \otimes \mathbb{Z}_2 / Ker\phi \cong \mathbb{Z}_4 \otimes \mathbb{Z}_4.$$

Comparing the number of elements we get that $|Ker\phi| = 2$. Since $\mathbb{Z}_{16} \otimes \mathbb{Z}_2$ is Abelian, all subgroups are normal. So possibilities for $Ker\phi$ are subgroups of order 2 in $\mathbb{Z}_{16} \otimes \mathbb{Z}_2$. There are three such subgroups:

$H_1 = \{(0,0),(8,1)\}$, $H_2 = \{(0,0),(0,1)\}$, and $H_3 = \{(0,0),(8,0)\}$.

If $Ker\phi = H_1$ then $(1,0) + Ker\phi$ has order 16 but $\mathbb{Z}_4 \otimes \mathbb{Z}_4$ has no element of order 16.
If $Ker\phi = H_2$ then $(1,0) + Ker\phi$ has order 16 but $\mathbb{Z}_4 \otimes \mathbb{Z}_4$ has no element of order 16.
If $Ker\phi = H_3$ then $(1,0) + Ker\phi$ has order 8 but $\mathbb{Z}_4 \otimes \mathbb{Z}_4$ has no element of order 8.
Therefore there are no homomorphisms from $\mathbb{Z}_{16} \otimes \mathbb{Z}_2$ onto $\mathbb{Z}_4 \otimes \mathbb{Z}_4$.

Chapter 10 #20. There are no homomorphisms from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_8$ because if there were such a homomorphism, say ϕ, then by the first isomorphism theorem $Z_{20}/Ker\phi \cong Z_8$. This would imply $20 = |Ker\phi| \times 8$(why?) which is not possible.

There are homomorphisms from $\mathbb{Z}_{20} to Z_8$ that are not onto.
Let ϕ be one such map. Then $\mathbb{Z}_{20}/Ker\phi \cong Im\phi$. Since $\mathbb{Z}_{20}$ is finite, we have $|Ker\phi||Im\phi| = 20$

$Ker\phi$ is a subgroup of $\mathbb{Z}_{20}$ and $Im\phi$ is a subgroup of $\mathbb{Z}_8$ so possibilities are
1. $|Ker\phi| = 5$ and $|Im\phi| = 4$
2. $|Ker\phi| = 10$ and $|Im\phi| = 2$

Recall that Z_{20} has only one subgroup of order 5, and only one subgroup of order 10.
Because Z_{20} is cyclic ϕ is determined by its action on its generator 1.

For case 1, i.e. when $|Ker\phi| = 5$, we have $Ker\phi = \{0,4,8,12,16\}$.
Let $\phi(1) = 2$ (it should map to something of order 4 in Z_8, why?).

All the elements in a coset should map to the same element (why?), so
elements in $0 + Ker\phi = \{0,4,8,12,16\}$ map to 0.
elements in $1 + Ker\phi = \{1,5,9,13,17\}$ map to 2.
elements in $2 + Ker\phi = \{2,6,10,14,18\}$ map to 4. (why?)
elements in $3 + Ker\phi = \{3,7,11,15,19\}$ map to 6. (why?)

Carry out a similar argument for the second case, that is when $|Ker\phi| = 10$.

Chapter 10 #33. $\phi : U_{40} \to U_{40}$ is a homomorphism with kernel $\{1,9,17,33\}$. Suppose $\phi(11) = 11$

Note that for a homomorphism ϕ, $\phi(a) = \phi(b)$, iff $\phi(ab^{-1}) = 1$ iff $ab^{-1} \in Ker\phi$, that is $a \in bKer\phi$. Therefore if $\phi(11) = 11$ then the elements that map to 11 are precisely the elements in $11Ker\phi = \{11,19,27,3\}$.

Chapter 10 #34. Find a homomorphism as in problem 33.
We need to determine where each of the cosets of $Ker\phi$ map. We already have
elements in $11Ker\phi = \{1,9,17,33\}$ map to 1.
elements in $11Ker\phi = \{11,19,27,3\}$ map to 11.

If we map the other two cosets as follows we get a homomorphism as desired:
Elements in $7Ker\phi = \{7, 23, 39, 31\}$ map to 7.
Elements in $13Ker\phi = \{13, 37, 21, 29\}$ map to 13.

Think: Can you find others?

Chapter 10 #36. Let $\phi : \mathbb{Z} \otimes Z \to G$ be such that $\phi((3,2)) = a$ and $\phi((2,1)) = b$.
Then $\phi((3,2) - (2,1)) = \phi((3,2)) - \phi((2,1)) = a - b$ and $\phi((4,4)) = \phi(4(1,1)) = 4\phi((1,1) = 4(a - b)$ using the fact that ϕ is a homomorphism.

Chapter 10 #39. Second Isomorphism Theorem: If K is a subgroup of G and N is a normal subgroup of G, then $K/(K \cap N) \cong KN/N$.

1. Prove that $K \cap N$ is normal in K.
2. Prove that N is normal in KN.
3. Prove that the map $\phi : K \to KN/N$ given by $\phi(k) = kN$ is a homomorphism and is onto.
4. Apply the first isomorphism theorem.

Proofs:
1. Let $a \in K \cap N$ and $b \in K$. We need to prove that $x = bab^{-1} \in K \cap N$. Since N is normal in G and $b \in G$ and $a \in N$ we have $x \in N$.
x is also in K because a and b are both in K and K is a subgroup.

2.Let $a \in N$ and $b \in KN$. We need to prove that $x = bab^{-1} \in N$. Since N is normal in G and $b \in KN \subset G$ and $a \in N$ we have $x \in N$.

3.Let $k_1, k_2 \in K$. Then $\phi(k_1k_2) = (k_1k_2)N = k_1Nk_2N = phi(k_1)\phi(k_2)$. The second equality holds by definition of product of cosets. Therefore ϕ is a homomorphism.

ϕ is onto because a coset in KN/N is of the form knN for some $k \in K$ and $n \in N$. Note that $nN = N$ because $n \in N$. Then $\phi(K) = kN$ so any coset coset in KN/N is hit by ϕ.

4. $Ker\phi = \{k \in K | \phi(k) = N\}$ so $Ker\phi = \{k \in K | kN = N\} = \{k \in K | k \in N\} = K \cap N$.
Hence by the first isomorphism theorem $K/(K \cap N) \cong KN/N$

Chapter 10 #50. Let $G = \langle a \rangle$. Any element in G is of the form a^k for some $k \in \mathbb{Z}$ and for a homomorphism ϕ, $\phi(a^k) = (\phi(a))^k$. So if $\phi(a)$ is given, the images of the rest of the elements will be determined by it.