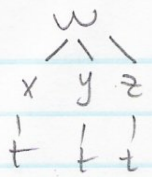
MATH 250 Homework 13 Solutions

Sec 13.5 #10. $w = xyz$, $x = t^2$, $y = 2t$, $z = e^{-t}$

a)



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= yz\cdot 2t + xz\cdot 2 + xy\cdot(-e^{-t})$$

b) $w = t^2 \cdot 2t \cdot e^{-t} = 2t^3 e^{-t} \Rightarrow \frac{dw}{dt} = 2\cdot(3t^2e^{-t} - t^3e^{-t})$

$$= 6e^{-t}t^2 - 2e^{-t}t^3$$

#12. $x_1 = 48\sqrt{2}\,t$ $\qquad$ $x_2 = 48\sqrt{3}\,t$

$y_1 = 48\sqrt{2}t - 16t^2$ $\qquad$ $y_2 = 48t - 16t^2$

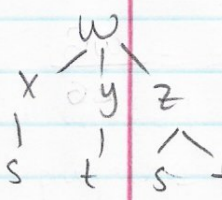
distance between the two particles $= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$$f(t) = \sqrt{(48\sqrt{2}t - 48\sqrt{3}t)^2 + \left[48\sqrt{2}\,t - 16t^2 - (48t - 16t^2)\right]^2}$$

$$= 48t\sqrt{(\sqrt{2}-\sqrt{3})^2 + (\sqrt{2}-1)^2} = 48t\sqrt{8 - 2\sqrt{6} - 2\sqrt{2}}$$

$$f'(t) = 48\cdot\sqrt{8-2\sqrt{6}-2\sqrt{2}} \cong 25.06$$

#24. $w = x\cos(yz)$, $\quad x = s^2$, $\quad y = t^2$, $z = s - 2t$



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s} = \cos(yz)\cdot 2s - xy\sin(yz)\cdot 1$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t} = -xz\sin(yz)\cdot 2t + 2xy\sin(yz)$$

#32. $xz + yz + xy = 0$

To find $\frac{\partial z}{\partial x}$ differentiate both sides with respect to x keeping y as a constant.

product rule

$$\left(x\cdot\frac{\partial z}{\partial x} + 1\cdot z\right) + y\cdot\frac{\partial z}{\partial x} + y\cdot 1 = 0$$

$$(x+y)\frac{\partial z}{\partial x} = -z - y \Rightarrow \frac{\partial z}{\partial x} = -\frac{(z+y)}{(x+y)}$$

Similar for $\frac{\partial z}{\partial y} = -\frac{x+z}{(x+y)}$

Sec 13.6 #16. $g(x,y) = xe^y$ $\theta = \frac{2\pi}{3} \Rightarrow u = \langle \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3} \rangle = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$g_x = e^y$, $g_y = xe^y$ $\Rightarrow$ $D_u\, g(x,y) = -\frac{1}{2} \cdot e^y + \frac{\sqrt{3}}{2} xe^y$

#18. $f(x,y) = \cos(x+y)$ @ $P(0,\pi)$ towards $Q(\frac{\pi}{2}, 0)$

$f_x = -\sin(x+y)$ @ $P(0,\pi) \Rightarrow f_x = -\sin\pi = 0$

$f_y = -\sin(x+y)$ @ $P(0,\pi) \Rightarrow f_y = -\sin\pi = 0$

$\Downarrow$ direction $\langle \frac{\pi}{2} - 0, 0 - \pi \rangle$

Because f_x and f_y are both zero the directional derivative is zero at $(0,\pi)$.

#34. $g(x,y) = ye^{-x^2}$ $P(0,5)$ $g_x = -2xye^{-x^2}\Big|_{(0,5)} = 0$

$\nabla g(0,5) = \langle 0, 1 \rangle$ $g_y = e^{-x^2}\Big|_{0,5} = 1$

Max value of directional derivative $\|\nabla g(0,5)\| = \|\langle 0,1 \rangle\| = 1$.

#48. $f(x,y) = 9 - x^2 - y^2$ $f_x = -2x$, $f_y = -2y$

$\nabla f(1,2) = \langle -2 \cdot 1, -2 \cdot 2 \rangle = \langle -2, -4 \rangle$

a) $u = \langle \cos\frac{-\pi}{4}, \sin\frac{-\pi}{4} \rangle = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$ $\Rightarrow D_u f(1,2) = -2 \cdot \frac{\sqrt{2}}{2} + -4 \cdot \left(\frac{-\sqrt{2}}{2}\right)$

$= -\sqrt{2} + 2\sqrt{2} = \sqrt{2}$

b) $u = \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle \Rightarrow D_u f(1,2) = -2 \cdot \frac{1}{2} + -4 \cdot \frac{\sqrt{3}}{2} = -1 - 2\sqrt{3}$

#50. $u \perp \nabla f(1,2) \Rightarrow u \cdot \nabla f(1,2) = 0 \Rightarrow$ let $u = \langle a, b \rangle$

Then $a^2 + b^2 = 1$ and $-2a - 4b = 0$.

Solving them together $a = \frac{2}{\sqrt{5}}$, $b = \frac{1}{\sqrt{5}}$

In the direction of $\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$ the directional derivative is zero that is the function does not change much.

#54. $f(x,y) = \dfrac{8y}{1 + x^2 + y^2}$

a) Level curve at $c = 2$

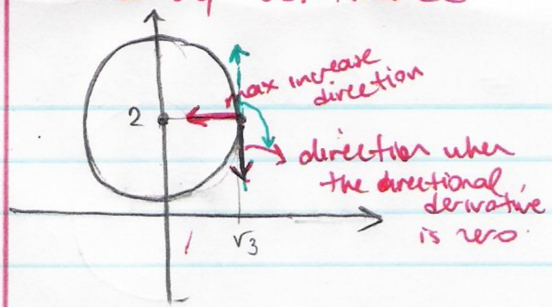
$2 = \dfrac{{}^4 8y}{1 + x^2 + y^2} \Rightarrow x^2 + y^2 - 4y = -1$

circle $\leftarrow x^2 + (y-2)^2 = 3$

b) Greatest increase is in the direction of $\nabla f(\sqrt{3}, 2) = \langle -\frac{\sqrt{3}}{2}, 0 \rangle$

(find $\langle f_x, f_y \rangle$ and plug in $x = \sqrt{3}$, $y = 2$)

Sec 13.6 #54 continued



c) Directional derivative is zero when $\vec{u}$ is $\perp$ to $\nabla f(\sqrt{3},2)$

So, $\vec{u} \cdot \langle -\frac{\sqrt{3}}{2}, 0 \rangle = 0$

$\langle a,b \rangle \cdot \langle -\frac{\sqrt{3}}{2}, 0 \rangle = 0$

$a \cdot \left(-\frac{\sqrt{3}}{2}\right) + 0 = 0 \Rightarrow a = 0$, $b \in \mathbb{R}$

d) see the notebook

$\vec{u} = \langle 0, 1 \rangle$ (unit vector)

or $\vec{u} = \langle 0, -1 \rangle$

Sec 13.7 #22. $z = x^2 - 2xy + y^2$ @ $(1,2,1)$

$F(x,y,z) = x^2 - 2xy + y^2 - z = 0$

$F_x = 2x - 2y$; $F_y = -2x + 2y$, $F_z = -1$

@ $(1,2,1)$ $(-2, 2, -1) \rightarrow$ normal

tangent plane: $-2(x-1) + 2(y-2) - (z-1) = 0$

$-2x + 2y - z - 1 = 0$

#34. $xyz = 10$

$F(x,y,z) = xyz - 10 = 0 \Rightarrow F_x = yz \quad F_y = xz \quad F_z = xy$

@ $(1,2,5)$ $\langle 10, 5, 2 \rangle$ normal

tangent plane: $10(x-1) + 5(y-2) + 2(z-5) = 0$

$\boxed{10x + 5y + 2z - 30 = 0}$

normal line $\boxed{\frac{x-1}{10} = \frac{y-2}{5} = \frac{z-5}{2}}$

#46. $z = x^2 + y^2 \Rightarrow F(x,y,z) = x^2 + y^2 - z = 0$

$x + y + 6z = 33 \Rightarrow G(x,y,z) = x + y + 6z - 33 = 0$

$P(1,2,5)$

$\nabla F = \langle 2x, 2y, -1 \rangle \Big|_{(1,2,5)} = \langle 2, 4, -1 \rangle$

$\nabla G = \langle 1, 1, 6 \rangle \Big|_{(1,2,5)} = \langle 1, 1, 6 \rangle$

$\langle 2,4,-1 \rangle \times \langle 1,1,6 \rangle = \langle 25, -13, -2 \rangle$

direction of the tangent line to the curve of intersection. Why?

a) equation of the tangent line: $\frac{x-1}{25} = \frac{y-2}{-13} = \frac{z-5}{-2}$

b) Angle $= \theta \Rightarrow \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \, \|G\|} = 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$