Sec 12.2 #80. Prove: If $\vec{r}(t)\cdot\vec{r}(t)$ is constant then $\vec{r}(t)\cdot\vec{r}\,'(t)=0$

Let $\vec{r}(t)=\langle f(t), g(t), h(t)\rangle = \langle f, g, h\rangle$

$\vec{r}\cdot\vec{r}$ constant $\Rightarrow f^2+g^2+h^2 = \text{constant} = c$

Differentiate with respect to t (Chain Rule!) $\quad 2ff' + 2gg' + 2hh' = 0$

$$2(ff'+gg'+hh')=0$$
$$ff'+gg'+hh'=0$$
$$\langle f,g,h\rangle\cdot\langle f',g',h'\rangle=0$$
$$\vec{r}\cdot\vec{r}\,'=0$$

Sec 12.3 #26. Projectile: $h_0=3\text{ft}$, $\|v_0\|=900\text{ft/sec}$ $\theta=45°$

$$\vec{r}(t)=900(\cos 45)t\,\vec{i}+\left(900(\sin 45)t+3-\tfrac{1}{2}32\cdot t^2\right)\vec{j}$$
$$\vec{r}(t)=450\sqrt{2}\,t\,\vec{i}+\left(450\sqrt{2}\,t+3-16t^2\right)\vec{j}$$

range: when $450\sqrt{2}t+3-16t^2=0 \Rightarrow t=\dfrac{-450\sqrt{2}\mp\sqrt{405192}}{-32}$

$\Rightarrow t\cong 39.779$

range $=450\sqrt{2}\,(39.779)\cong\boxed{25315.5 \text{ feet}}$

max height when $\vec{j}$ component of velocity is zero

$$v(t)=450\sqrt{2}\,i+\left(\underbrace{450\sqrt{2}-32t}\right)\vec{j}$$

0 when $t=\dfrac{450\sqrt{2}}{32}$

max height $=\vec{j}$ component of the position $=450\sqrt{2}\left(\dfrac{450\sqrt{2}}{32}\right)+3-16\cdot\left(\dfrac{450\sqrt{2}}{32}\right)^2$

$=\boxed{6331.125\text{ft}}$

#42. $\vec{r}(t)=V_0(\cos 8)t\,i+\left(V_0(\sin 8)t-4.9t^2\right)\vec{j}$

range $=50$ m $=$ x-coordinate when y-coordinate is zero.

So, $V_0(\sin 8)t-4.9t^2=0 \Rightarrow \begin{cases} t=0 \\ t=\dfrac{V_0\sin 8}{4.9}\end{cases}$

Then $V_0\cos(8)\cdot\dfrac{V_0\sin 8}{4.9}=50 \Rightarrow \boxed{V_0=42.2 \text{ m/sec}}$ speed.

Sec 12.3 #45–48. and Sec 12.4 #49–48

45. $\mathbf{v}(t) = -b\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$

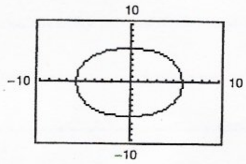
$\mathbf{r}(t) \cdot \mathbf{v}(t) = -b^2\omega \sin(\omega t)\cos(\omega t) + b^2\omega \sin(\omega t)\cos(\omega t) = 0$

Therefore, $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are orthogonal.

46. (a) Speed $= \|\mathbf{v}\| = \sqrt{b^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)}$

$= \sqrt{b^2\omega^2[\sin^2(\omega t) + \cos^2(\omega t)]} = b\omega$

(b)



The graphing utility draws the circle faster for greater values of ω.

47. $\mathbf{a}(t) = -b\omega^2 \cos(\omega t)\mathbf{i} - b\omega^2 \sin(\omega t)\mathbf{j} = -b\omega^2[\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}] = -\omega^2\mathbf{r}(t)$

$\mathbf{a}(t)$ is a negative multiple of a unit vector from $(0, 0)$ to $(\cos \omega t, \sin \omega t)$ and thus $\mathbf{a}(t)$ is directed toward the origin.

48. $\|\mathbf{a}(t)\| = b\omega^2\|\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\| = b\omega^2$

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45. $\mathbf{r}(t) = a \cos \omega t\, \mathbf{i} + a \sin \omega t\, \mathbf{j}$

$\mathbf{v}(t) = -a\omega \sin \omega t\, \mathbf{i} + a\omega \cos \omega t\, \mathbf{j}$

$\mathbf{a}(t) = -a\omega^2 \cos \omega t\, \mathbf{i} - a\omega^2 \sin \omega t\, \mathbf{j}$

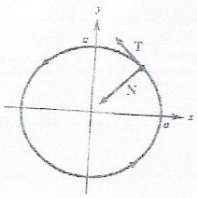
$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = -\sin \omega t\, \mathbf{i} + \cos \omega t\, \mathbf{j}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos \omega t\, \mathbf{i} - \sin \omega t\, \mathbf{j}$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = a\omega^2$

46. $\mathbf{T}(t)$ points in the direction that $\mathbf{r}$ is moving. $\mathbf{N}(t)$ points in the direction that $\mathbf{r}$ is turning, toward the concave side of the curve.



47. Speed: $\|\mathbf{v}(t)\| = a\omega$

The speed is constant since $a_{\mathbf{T}} = 0$.

48. If the angular velocity ω is halved,

$$a_{\mathbf{N}} = a\left(\frac{\omega}{2}\right)^2 = \frac{a\omega^2}{4}.$$

$a_{\mathbf{N}}$ is changed by a factor of $\frac{1}{4}$.

Sec 12.4 #16. $\vec{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t \rangle$ $P(1, \sqrt{3}, 1) \Downarrow t = \frac{\pi}{6}$

$$T(t) = \frac{\vec{r}\,'(t)}{\|r'(t)\|} = \frac{\langle 2\cos t, -2\sin t, 8\sin t\cos t \rangle}{\sqrt{4\cos^2 t + 4\sin^2 t + 8^2\sin^2 t\cos^2 t}}$$

$$@\ t = \frac{\pi}{6},\ T\left(\frac{\pi}{6}\right) = \frac{\langle \sqrt{3}, -1, 2\sqrt{3} \rangle}{\sqrt{4 + 64\cdot\frac{1}{4}\cdot\frac{3}{4}}} = \left\langle \frac{\sqrt{3}}{4}, \frac{-1}{4}, \frac{\sqrt{3}}{2} \right\rangle$$

direction $= \langle \sqrt{3}, -1, 2\sqrt{3} \rangle$

tangent line: $\begin{pmatrix} x = 1 + \sqrt{3}t \\ y = \sqrt{3} - t \\ z = 1 + 2\sqrt{3}t \end{pmatrix}$

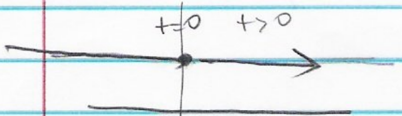
#30. $r(t) = \langle \cos t, 2\sin t, 1 \rangle$ $\vec{N}\ @\ t = \frac{-\pi}{4}$

$$T(t) = \frac{\langle -\sin t, 2\cos t, 0 \rangle}{\sqrt{\sin^2 t + 4\cos^2 t}} = \frac{\langle -\sin t, 2\cos t, 0 \rangle}{\sqrt{1 + 3\cos^2 t}}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \left(\frac{1}{2\sqrt{1+3\cos^2 t}} \cdot 2\cos t\cdot(-\sin t) \langle -\sin t, 2\cos t, 0 \rangle + \frac{1}{\sqrt{1+3\cos^2 t}} \langle -\cos t, -2\sin t, 0 \rangle \right) \cdot \frac{1}{\|T'\|}$$

plug in $t = \frac{-\pi}{4} \Rightarrow N\left(\frac{-\pi}{4}\right) = \left\langle \frac{-2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, 0 \right\rangle$ → note that it is a unit vector.

#34. $\vec{r}(t) = \langle 0, t^2, 1 \rangle \Rightarrow \vec{T}(t) = \frac{r'}{\|r\|} = \frac{\langle 0, 2t, 0 \rangle}{\sqrt{4t^2}} = \langle 0, 1, 0 \rangle$ when $t > 0$.



$\vec{N}(t) = \frac{T'(t)}{\|T'(t)\|}$ ~~undefined~~ undefined

The path is a line and the speed varies by t

#54. $r(t) = 4t i - 4t j + 2t k$ @ $t = 2$.

$v(t) = 4i - 4j + 2k \to a(t) = 0$

$T(t) = \frac{v}{\|v\|} = \frac{1}{3}(2i - 2j + k)$ $N(t) = \frac{T'(t)}{\|T'(t)\|}$ is undefined

Then a_T and a_N are not defined